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FLOW PAST A STRONGLY HEATED SPHERE BY A GAS WITH LOW REYNOLDS NUMBERS

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If the relative drop of temperature in slow flows past solids is of the order of unity, then the Barnett temperature stresses appearing in the impulse equation are, for the Reynolds numbers $R_\infty \leqslant 1$, of the order of the usual viscous stresses [1, 2]. The problem of a gas flow past an uniformly heated (cooled) sphere with the above-mentioned temperature stresses taken into account and $R_\infty \ll 1$, is solved for the case when the effect of the gravitational convection is inessential. The temperature stresses lead to considerable reduction in the drag of the sphere compared with the value obtained with the help of the Navier-Stokes equations [3].

Slow $(M_\infty \ll 1)$ flows past uniformly heated (cooled) bodies with $R_\infty \leqslant O(1)$ are described by the following dimensionless equations of conservation and the boundary conditions [1]

$$\nabla \mathbf{v} = (\mathbf{v} \cdot \nabla \ln T) \tag{1}$$

$$^2/_3 (\mathbf{v} \cdot \nabla \ln T) = \nabla (T \nabla T) \tag{2}$$

$$T^{-1} (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla \Pi = \boldsymbol{\Pi}^{(1)} - {}^3/_2 \delta (\nabla T)^2 \nabla T + 2\delta (\mathbf{v} \cdot \nabla \ln T) \nabla T - {}^3/_2 \delta \nabla [T (\nabla T)^2] \tag{3}$$

$$\Pi = \pi + \frac{4\delta - 1}{3} (\mathbf{v} \cdot \nabla T), \quad \Pi_i^{(1)} = \frac{\partial}{\partial x_j} T \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \delta_{ij} \nabla \mathbf{v} \right) \tag{4}$$

It is assumed that the external forces can be neglected. Following [1] it can be shown that the influence of the gravitational convection is inessential in the case of a sphere provided that $L^3 < \nu_\infty^2 R_\infty / g$ where g is acceleration due to gravity. Under the normal conditions and $R_\infty \sim 10^{-1}$ to 10^{-2} we have practically $L < 10^{-2}$ cm. The above equations hold with the accuracy of the order $O(K^2)$, where K is the Knudsen number. In an unperturbed flow we have

$$\mathbf{v} = R_\infty \mathbf{e}, \qquad T = 1$$
$$R_\infty = u_\infty L / \nu_\infty, \quad \nu_\infty = \mu_\infty / \rho_\infty, \quad \mathbf{e} = \mathbf{v} / v \tag{5}$$

At the surface of the body we have

$$\mathbf{v} = 0, \quad T = T_w \tag{6}$$

with the accuracy of the order $O(K)$ (this is the order of the velocity and temperature jumps at the walls).

The stream velocity is $\mathbf{u} = V \cdot \mathbf{v}$ and the "viscous" velocity is [1] $V = \nu_\infty / L$. The coordinates $(x_1, x_2, x_3) = (x, y, z)$ are relative to the characteristic dimension L; the temperature T, pressure p, etc. are related to their values in the unperturbed flow, therefore

$$p = 1 + (\rho_\infty V^2 / p_\infty)\, \pi$$

i.e. π is the dimensionless variable part of the pressure. The density ρ is eliminated using the equation of state.

Here and below we consider a monatomic gas of Maxwellian molecules, when the dimensionless coefficient of viscosity is $\mu = T$ and the ratio of specific heats and the Prandtl number are $^5/_3$ and $^2/_3$, respectively. The coefficient $\delta = 1$ is introduced to single out the Barnett terms of the impulse equation. Expressions for Π and $\Pi_i^{(1)}$ are written in a form somewhat different from that in [1], and more suitable for comparing our solution with the solution obtained within the framework of the Navier-Stokes equations (we note that at the body surface $\Pi = \pi$). The navier-Stokes equations hold ($\delta = 0$) if for a fixed R_∞ the quantity $T_w \to 1$.

The variable part of the stress tensor at the wall, with the no-slip conditions and Eqs. (1) and (2) taken into account, has the form

$$\Pi_{ij} = \pi\delta_{ij} - T\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{3}{2}\,\delta T\,\frac{\partial^2 T^2}{\partial x_i \partial x_j} \qquad (7)$$

If $R_\infty = 0$ (i.e. $u_\infty = 0$), then we generally have a thermally stressed convective flow at the body, with velocities (*) of the order of V (at $T_w - 1 = O(1)$), the convection governed by the Barnett temperature stresses and described by the complete system of equations (1) – (4). This convection may lead to appearance of a "thermal stress" force acting on the body.

When $R_\infty \ll 1$, the incoming flow has only a perturbing effect and the solution should be sought in the form

$$\mathbf{v} = \mathbf{v}_0 + \varepsilon\mathbf{v}_1 + \ldots, \quad T = T_0 + \varepsilon T_1 + \ldots, \quad \pi = \pi_0 + \varepsilon\pi_1 + \ldots, \quad \varepsilon = R_\infty \qquad (8)$$

The flow past a sphere of radius $r_0 = L$ is an exception. In this case the gas is at rest when $u_\infty = 0$ [2], i.e. $\mathbf{v}_0 = 0$ and the energy equation (2) yields the Laplace's equation for T_0^2, from which we obtain

$$T_0 = (1 + \omega / r)^{1/2}, \quad \omega = T_w^2 - 1 \qquad (9)$$

The temperature stresses lead only to redistribution of the pressure π_0 along r (the length r of the radius vector is counted from the center of the sphere and below we use a spherical (r, θ, φ) -coordinate system). For this reason, when the flow past a sphere has the Reynolds number $R_\infty \ll 1$, the characteristic velocity is not V but u_∞.

Let us replace $\mathbf{v}_1$ by the gas velocity $\mathbf{u}$ relative to u_∞. Then Eqs. (1), (3) and (4) become

$$\nabla \mathbf{u} = (\mathbf{u}\cdot\nabla \ln T_0) + O(R_\infty) \qquad (10)$$

$$R_\infty T_0^{-1}(\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla\Pi_1 = \mathbf{\Pi}_1^{(1)} + 2\delta\,(\mathbf{u}\cdot\nabla \ln T_0)\,\nabla T_0 - {}^3/_2\delta\,\{(\nabla T_0)^2\,\nabla T_1 + 2\,(\nabla T_0\cdot\nabla T_1)\nabla T_0 + \nabla\,[2T_0\,(\nabla T_0\cdot\nabla T_1) + T_1\,(\nabla T_0)^2]\} + O(R_\infty) \qquad (11)$$

$$\Pi_{1i}^{(1)} = \frac{\partial}{\partial x_j}\,T_0\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \delta_{ij}\,\frac{\partial u_k}{\partial x_k}\right) \qquad (12)$$

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*) If $T_w \gg 1$, then the conditions at the wall are typical [1], in particular we have $V = \nu_w / L$.

An expression for Π_1 is obtained in the analogous manner. When the temperature stresses are discarded ($\delta = 0$) and terms of the order $O(R_\infty)$ neglected, the system (10)–(12) describes, within the framework of the Navier-Stokes equations, the known problem [3] of a "Stokes" flow past a heated sphere. The procedure of matching the outer and inner asymptotic expansions [4] must be employed to obtain the further terms of expansion of the solution into a series in R_∞

In the case under consideration, the situation is different. The impulse equation includes T_1 and the energy equation in the first approximation must be solved simultaneously with (10) and (11). We obtain T_1 using the method of inner and outer expansions. The energy equation written in the (inner) variables introduced above, has the form

$$\frac{2}{3}\frac{R_\infty}{T}u_j\frac{\partial T}{\partial x_j} = \frac{\partial}{\partial x_j}T\frac{\partial T}{\partial x_j} \tag{13}$$

In the outer variables $\rho = R_\infty r$ and $X_i = R_\infty x_i$ it becomes

$$\frac{2}{3T}u_j^*\frac{\partial T^*}{\partial X_j} = \frac{\partial}{\partial X_j}T^*\frac{\partial T^*}{\partial X_j} \tag{14}$$

We denote for convenience the functions of the outer variables with an upper asterisk. The inner and the outer expansions have the form

$$T = T_0 + R_\infty T_1 + \ldots, \qquad \mathbf{u} = \mathbf{u}_0 + R_\infty \mathbf{u}_1 + \ldots \tag{15}$$

$$T^* = 1 + R_\infty T_1^* + \ldots, \qquad \mathbf{u}^* = \mathbf{e}_x + R_\infty \mathbf{u}_1^* \ldots \tag{16}$$

Here T_0 is given by (9), while T_1 and T_1^* satisfy the following equations:

$$\nabla^2 T_0 T_1 = \frac{2}{3T_0}u_{0r}\frac{dT_0}{dr} \tag{17}$$

$$\nabla^2 T_1^* = \frac{2}{3}\frac{\partial T_1^*}{\partial X} \tag{18}$$

The last equation is reduced to the Helmhotz equation

$$\nabla^2\Omega = {}^1/_9\Omega, \qquad \Omega = e^{-{}^1/_3 X}T_1^*$$

whose solutions are well known. Using the conditions of boundedness of the solution at $\rho \to \infty$ and the principle of minimum singularity [4] at $\rho \to 0$, we choose the following solution of this equation:

$$\Omega = C\rho^{-1}\exp(-{}^1/_3\rho)$$

where C is a function of R_∞. To find $T_1(r \to \infty, \theta)$ we match the two-term inner and outer expansions

$$T = \left(1 + \frac{\omega}{r}\right)^{1/2} + R_\infty T_1(r, \theta), \qquad T^* = 1 + R_\infty\frac{C}{\rho}\exp\frac{X-\rho}{3}$$

Applying the standard methods [4] we obtain $C = \omega / 2$ and

$$T_1(\infty, \theta) = {}^1/_6\,\omega(\cos\theta - 1) \tag{19}$$

Taking (19) into account, we seek the solution of the problem in the form

$$u_r = f(r)\cos\theta, \quad u_\theta = -g(r)\sin\theta, \quad \Pi_1 = h(r)\cos\theta + m(r)$$

$$T_1 = T_0^{-1}[\psi(r) + t(r, \theta)], \quad t = \tau(r)\cos\theta \tag{20}$$

with the boundary conditions

$$f = 0,\ g = 0,\ \psi = 0,\ \tau = 0 \qquad (r = 1)$$
$$f \to 1,\ g \to 1,\ \tau \to \omega / 6,\ \psi \to -\omega / 6 \qquad (r \to \infty) \tag{21}$$

Inserting (20) into (17) we see that the variables separate, $\psi = \omega (1 - r) / (6r)$ and τ satisfies the equation

$$\tau'' + \frac{2}{r}\tau' - \frac{2}{r^2}\tau - \frac{2}{3}\frac{T_0'}{T_0} f = 0 \tag{22}$$

The temperature stresses expressed in terms of ψ determine $m(r)$, i.e. they cause a redistribution of the pressure along r, therefore they are not taken into account in what follows. Neglecting in (10) and (11) the terms of the order of R_∞ as compared with unity (the convective terms in particular) and using (20), we find

$$f' + (2 / r)(f - g) - T_0^{-1} T_0' f = 0 \tag{23}$$

$$h' - T_0 \left[f'' + \frac{2}{r} f' - \frac{4}{r^2}(f - g)\right] - 2T_0' f' + T_0^{-1} T_0'^2 f +$$
$$3\delta T_0' \left[\tau'' + \frac{2}{r}(T_0^{-1} T_0' \tau - \tau') - \frac{2}{3} T_0^{-1} T_0' f\right] = 0 \tag{24}$$

$$-h + T_0 [rg'' + 2g' + 2r^{-1}(f - g)] + T_0' (rg' + f - g) - 3\delta T_0' \tau' = 0 \tag{25}$$

Thus the problem has been reduced to the solution of a linear system of ordinary differential equations (22) – (25) with the boundary conditions (21). For numerical integrations these equations can be reduced to a more convenient form, using new variables ξ and H introduced by the formulas

$$r = |\omega| e^{\xi}, \qquad H = rh / T_0$$

Let us now investigate the contribution of the local temperature stresses to the force F acting on the sphere. The third term of (7) can be written as

$$^3/_2 T_w (T_0^2)_{ij} + {}^3/_2 R_\infty [2T_w (\psi)_{ij} + T_1 (T_0^2)_{ij} + 2T_w (t)_{ij}] \tag{26}$$
$$(\ )_{ij} = \frac{\partial^2 (\ )}{\partial x_i \partial x_j}$$

The first two terms of (26) make no contribution to F and the third term is equal to zero ($T_1 = 0$ at the sphere). We further have

$$(t)_{rr} = \tau'' \cos\theta, \quad (t)_{r\theta} = -(\tau / r)' \sin\theta$$

The contribution of the local temperature stresses to F is zero. Indeed, it is proportion-

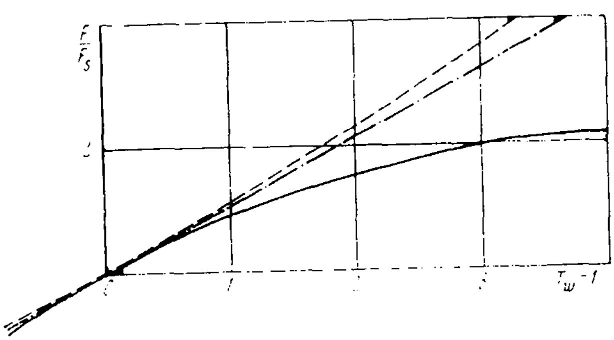

Fig. 1

al to

$$\int_0^{\pi} [(t)_{rr}\cos\theta - (t)_{r\theta}\sin\theta]\sin\theta\, d\theta = \left(\frac{\tau}{r}\right)' \int_0^{\pi} (1 - 3\cos^2\theta)\sin\theta\, d\theta = 0$$

Here we used Eq. (22) and the fact that at the wall $f = 0$. Thus we see that, as in the Navier-Stokes approximation, the force F acting on the sphere is an integral of the pressure and the viscous stresses over its surface, the latter however no longer defined by the Navier-Stokes equations

$$F = {}^2/_9\,(2g'T - h)_{r=1} F_S, \quad F_S = 6\pi\rho_\infty \mu_\infty r_0 \tag{27}$$

Here F_S is defined by the Stokes formula.

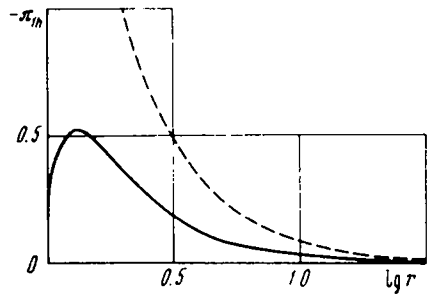

Fig. 2

Results of the computations utilizing Eqs. (22) – (25) have shown that the temperature stresses exert no appreciable influence on the velocity field, but lead to a sharp decrease in the value of F when T_w are large. The latter can be explained by a decrease (in absolute magnitude) in the value of the asymmetric component $\pi_{1h}(r)\cos\theta$ of the pressure where

$$\pi_{1h} = h - {}^1/_3\,(4\delta - 1)\, fT_0'.$$

Figures 1 and 2 show the results of computations with the temperature stresses taken into account ($\delta = 1$) using the solid lines and those without the temperature stresses ($\delta = 0$), using the broken lines. The dash-dot line in Fig. 1 was computed by the formula (4.6) of [2] derived by linearizing the solution, in the Navier-Stokes approximation with $(T_w - 1) \ll 1$ relative to the Stokes solution. The graphs in Fig. 2 are obtained for $T_w = 4$. When $r = 1$ and $\delta = 0$ the value of $\pi_{1h} = -3.66$.

Experimental data available [5] show that F increases with increasing T_w, the effect becoming more pronounced, the smaller R_∞ is. These data were, however, obtained for $R_\infty > 2$ under the conditions which made possible the manifestation of the effects of free gravitational convection.

BIBLIOGRAPHY

1. Galkin, V. S., Kogan, M. N. and Fridlender, O. G., On free convection in gas in the absence of external forces. Izv. Akad. Nauk SSSR, MZhG, №3, 1971.

2. Galkin, V. S., Kogan, M. N. and Fridlender, O. G., On certain kinetic effects in the flows of continuous medium. Izv. Akad. Nauk SSSR, MZhG, №3, 1970.

3. Kassoy, D. R., Adamson, T. C. and Messiter, A. F., Compressible low Reynolds Number flow around a sphere. Phys. Fluids, Vol. 9, №4, 1966.

4. Van Dyke, Milton, Perturbation Methods in Fluid Mechanics. Acad. Press, N.Y. and London, 1964.

5. Basina, I. P. and Maksimov, I. A., Influence of nonisothermal behavior on drag of a spherical particle. Coll.: Problems of thermoenergetics and applied thermophysics. №5, Alma-Ata, "Nauka", 1969.

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